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Technical Report No. 3

Prepared for Office of Naval Research Contract No. Nonr-988(11)

February 1963

PROJECT NR 083-16612-62
(Geophysics Branch ONR)

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1. Introduction:

Rattray (1959) developed a theory on internal tides caused by oncoming surface tides over a continental shelf on a rotating system. He considered the case of surface tides with wave lengths much larger than a width of the shelf which is assumed to have either a uniform or linearly increasing depth. The internal waves thus generated are standing waves over the shelf and progressive waves travelling seaward in the open sea. The amplitudes of the standing waves are large compared with those of surface tides and also with those of the progressive internal waves indicating spectral structure with maxima and minima according to frequencies.

However, internal waves which are observed over a continental shelf have sometimes periods different from tidal periods and also they propagate in directions other than normal to the coast. The present study deals with the waves travelling in two-dimensional directions in regards to the coast in a two-layered sea. Although the effect of Coriolis' force is also included, a wide range of periods of the waves is assumed.

2. Fundamental Equations:

In a two-layered rotating ocean, the x and y axis are taken perpendicular and parallel to the coast, respectively. The density difference between the two layers is taken as $\Delta\rho$ and the effect of friction is neglected. The equations of motion for the upper and lower layer are respectively: (with $\delta = \Delta\rho/\rho$)

$$\partial u'/\partial t - f v' = -g\,\partial \zeta'/\partial x \qquad (1)$$

$$\partial v'/\partial t + f u' = -g\,\partial \zeta'/\partial y \qquad (2)$$

and

$$\partial u''/\partial t - f v'' = -g\,(1-\delta)(\partial \zeta'/\partial x) - g\delta\,\partial \zeta''/\partial x \qquad (3)$$

$$\partial v''/\partial t + f u'' = -g\,(1-\delta)(\partial \zeta'/\partial y) - g\delta\,\partial \zeta''/\partial y \qquad (4)$$

The equation of continuity is, for the upper layer:

$$\partial(h'u')/\partial x + \partial(h'v')/\partial y = \partial \zeta''/\partial t - \partial \zeta'/\partial t \qquad (5)$$

and for the lower layer:

$$\partial(h''u'')/\partial x + \partial(h''v'')/\partial y = -\partial \zeta''/\partial t \qquad (6)$$

In these equations a primed quantity refers to the upper layer and a double-primed quantity to the lower layer; u and v are the horizontal components of velocity; ζ is the elevation of the upper surface of a layer; h is the layer depth; and g and f are a gravity constant and Coriolis' coefficient, respectively.

It is assumed that the depths h' and h'' are independent on y and that the quantities u, v and ζ are proportional to $e^{i(\sigma t + ky)}$.

Using u, v and ζ for the factors of the exponential term, the velocity components in the upper and lower layer are expresse by:

$$(\sigma^2 - f^2)\,u' = ig\left(\sigma \frac{d}{dx} + fk\right)\zeta' \qquad (7)$$

$$(\sigma^2 - f^2)\,v' = -g\left(f \frac{d}{dx} + \sigma k\right)\zeta' \qquad (8)$$

from equations (1) and (2) and

$$(\sigma^2 - f^2)\,u'' = i\left(\sigma \frac{d}{dx} + fk\right)(g''\zeta' + g'\zeta'') \qquad (9)$$

$$(\sigma^2 - f^2)\,v'' = -\left(f \frac{d}{dx} + \sigma k\right)(g''\zeta' + g'\zeta'') \qquad (10)$$

from equations (4) and (5), respectively, where

$$g' = g\delta \quad ; \quad g'' = g(1-\delta) \qquad (11)$$

Substituting (7) and (8) into (3), we have:

$$\left[\frac{d}{dx}\left\{h''\left(\sigma\frac{d}{dx}+fk\right)\right\}-kh''\left(f\frac{d}{dx}+\sigma k\right)\right](g\zeta'+g'\zeta'')=-\sigma(\sigma^2-f^2)\zeta'' \quad (12)$$

Similarly, from (9), (10) and (6), we have:

$$\left[\frac{d}{dx}\left\{h'\left(\sigma\frac{d}{dx}+fk\right)\right\}-kh'\left(f\frac{d}{dx}+\sigma k\right)\right](g\zeta')=\sigma(\sigma^2-f^2)(\zeta''-\zeta') \quad (13)$$

Elimination of ζ'' from (12) and (13) yields:

$$\sigma^2(\sigma^2-f^2)\zeta'+\sigma(\sigma^2-f^2)g\,\mathbb{P}(h)\zeta'+g\,g'\,\mathbb{P}(h')\,\mathbb{P}(h'')\zeta'=0 \quad (14)$$

where $h = h' + h''$ and $\mathbb{P}(H)$ is a differential operator defined by:

$$\mathbb{P}(H)=\frac{d}{dx}\left\{H\left(\sigma\frac{d}{dx}+fk\right)\right\}-Hk\left(f\frac{d}{dx}+\sigma k\right) \quad (15)$$

From this definition, there is a relation like

$$\mathbb{P}(h) = \mathbb{P}(h') + \mathbb{P}(h'') \quad (16)$$

3. Waves in the sea of a uniform depth:

When the depths h' and h'' are uniform, equation (14) can be separated into two modes; ζ_s' for barotropic mode and ζ_i' for baroclinic mode, such as:

$$(\sigma^2-f^2)\zeta_s' + gh\left(\frac{d^2}{dx^2}-k^2\right)\zeta_s' = 0 \quad (17)$$

$$(\sigma^2-f^2)\zeta_i' + g'(h'h''/h)\left(\frac{d^2}{dx^2}-k^2\right)\zeta_i' = 0 \quad (18)$$

The elevation of interface ζ'' can be expressed in terms of ζ' by use of (13). Substitution of (17) and (18) into (13) yields:

$$\zeta_s'' = (h''/h)\zeta_s' \quad (19)$$

$$\zeta_i'' = -\epsilon^{-1}\zeta_i' \quad (20)$$

where ϵ is a small constant of order of magnitude 10^{-3} since it is defined by:

$$\epsilon = \{(h/h'')\delta^{-1}-1\}^{-1} \approx \delta(h''/h) \quad (21)$$

The velocity components u', v' and u'', v'' can be determined in terms of ζ_s' or ζ_i' from equations (7) to (10). For the barotropic mode equation (19) indicates that ζ_s' and ζ_s'' are of an equal order of magnitudes. Therefore, the velocity components of the upper and lower layer of this mode have relations:

$$u_s' \approx u_s'' \quad ; \quad v_s' \approx v_s'' \tag{22}$$

since $g'' \approx g$ and $g' \ll g$. Expressing the right-hand side of (9) and (10) with ζ_i' by substitution of (20), we have:

$$h'' u_i'' = -h' u_i' \quad ; \quad h'' v_i'' = -h' v_i' \tag{23}$$

Therefore, elevation of the interface and velocity components of the lower layer are expressed by corresponding quantities of the upper layer. However, since equation (20) with (21) indicates that ζ_i' is about 10^{-3} times ζ_i'', it is convenient to express elevations and velocity components of the baroclinic modes with ζ_i''. Also hereafter ζ', ζ'' is sometimes called surface waves (do not confuse with those in a sense of deep sea waves) and internal waves, respective[illegible]

If a wave number of y direction is assumed to be k, the elementary solutions of (17) and (18) can be expressed by

$$\zeta_s' = \exp(\pm imx + iky + i\sigma t) \tag{24}$$

$$\zeta_i' = \exp(\pm inx + iky + i\sigma t) \tag{25}$$

in which

$$k^2 + m^2 = (\sigma^2 - f^2)(gh)^{-1} \; (= k_m^2) \tag{26}$$

$$k^2 + n^2 = (\sigma^2 - f^2)\, h\, (g'h'h'')^{-1} \; (= k_n^2) \tag{27}$$

When m or n is real, the elementary solutions (24) and (25) represent the plane waves. The directions in which those waves propagate make an angle of $\pm \tan^{-1}(m/k)$ or $\pm \tan^{-1}(n/k)$, respectively, clockwise with the negative y-direction. The wave number k_m or k_n measured along the direction of propagation of surface and internal waves is equal to the square root of the right hand side of (26) or (27) respectively.

The plane waves are possible only when $\sigma > f$. In Table 1 the wave numbers and corresponding wave lengths measured along the direction of propagation are shown for waves with periods of half day and one hours, respectively, in the sea of 30° latitudes. Two kinds of depths of two layers are assumed, corresponding to the continental shelf and the open sea. It is seen that the wave lengths of internal waves are much shorter than those of surface waves.

When $\sigma < f$, the right hand side of (26) and (27) becomes negative and thus, m and n are always imaginary when k is assumed to be real. In case of $\sigma > f$, when k exceeds k_m or k_n, m or n becomes imaginary.

If it is assumed that the sea consists of the semi-infinite plane of positive x to the right of the coastline of y-axis, the elementary solutions which are finite in the infinite distance should have a negative coefficient for x when m or n becomes imaginary. These solutions represent the waves called Kelvin waves which propagate along the coast with amplitudes decreasing exponentially from the coast (Lamb, 1945 p. 319).

In presence of the coastline on the y-axis, two solutions corresponding to double signs in (24) or (24) with real m or n represent the incident and reflected waves propagating obliquely with the coast.

In case of $\sigma > f$, three combinations of different types of waves in the surface and internal mode are possible, according to the values of k. For a range of k satisfying $k < k_m$, both surface and internal waves are plane waves. For a range $k_n > k > k_m$ the surface waves are of Kelvin wave type, while the internal waves are plane waves. For $k > k_n$, both are Kelvin waves. Only the last combination is possible for the case of $\sigma < f$.

In case of both waves being plane waves, the propagating direction of the internal waves is closer to the x-axis compared to the direction of the surface waves since $m < n$ for the same value of k . It is easily seen that the surface Kelvin waves can generate plane internal waves moving away from the coast for $k_n > k > k_m$. The values of k_m and k_n for the open sea listed in Table 1 indicate that the waves of tidal periods most likely have this range of values of k .

The velocity of Kelvin waves can be determined from a boundary condition at the coast. Vanishment of the velocity component normal to the coast determines the specific value of k for each of barotropic and baroclinic modes, for which velocities of Kelvin waves become $(gh)^{\frac{1}{2}}$ and $(g'h'h''h^{-1})^{\frac{1}{2}}$, respectively. It must be noted that these velocities are independent on the rate of rotation of the earth. On the other hand, the propagating velocity of plane waves depends on the Coriolis coefficient as seen in the equations for k_m and k_n of (26) and (27).

TABLE 1. Wave Numbers and Wave Lengths of Plane Waves

	SHELF		OPEN SEA	
Depth	h'(M) 50	h"(M) 150	h'(M) 50	h'(M) 150
Period	1/2 day	1 hour	1/2 day	1 hour
$k_m(KM^{-1})$	2.86×10^{-3}	3.96×10^{-2}	0.91×10^{-3}	1.26×10^{-2}
$2\pi/k_m(KM)$	2200	154	6900	498
$k_n(KM^{-1})$	0.146	2.02	0.128	1.77
$2\pi/k_n(KM)$	43	3.1	49	3.6

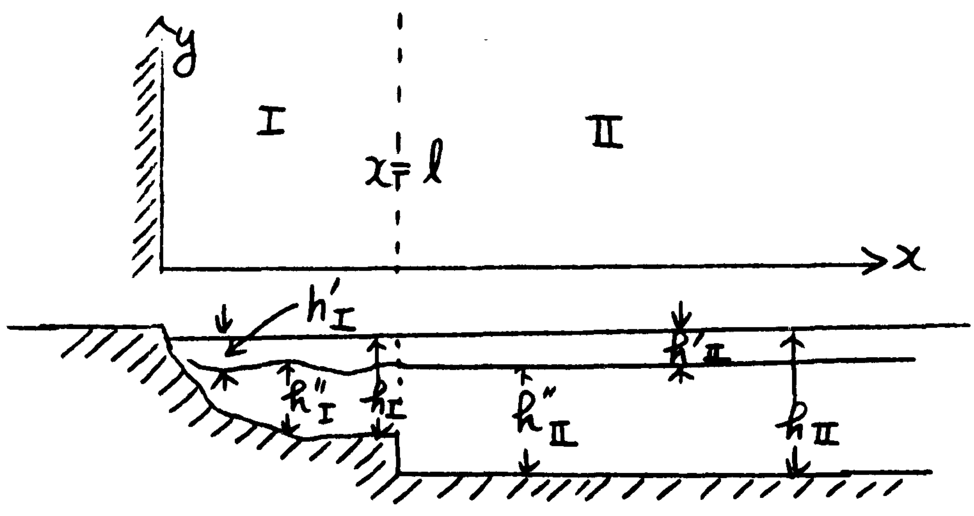
Note: M = meters

KM = kilometers

$f = 7.3 \times 10^{-4}\ (sec^{-1})$

$\delta = 2 \times 10^{-3}$

FIGURE 1. Schematical Configuration of the Ocean with a Shelf



4. General mathematical problems on waves over a shelf.

It is assumed that a continental shelf consisting of two layers with arbitrary depth distributions is bounded by the straight coast at $x = 0$ and by the straight edge at $x = \ell$, from where the open sea with two layers each of a uniform depth extends infinitely to the direction of positive x-axis. The quantities related to the shelf and the open sea are distinguished by suffices I and II, respectively, if such is necessary. (See Figure 1)

As an example, a mathematical problem on waves over the shelf in the barotropic mode (or for a uniform density) is considered. In this case, equation (14) which must be satisfied over the shelf becomes the second order ordinary differential equation with x, since $g' = 0$. Therefore, it has two elementary solutions. In the open sea the surface elevation must satisfy equation (17) only, which also has two elementary solutions. When the waves in the open sea are of plane wave type, that is, m and k are real, these two solutions represent respectively incident and reflected waves. Four unknown coefficients of elementary solutions on the shelf and in the open sea can be determined except for a common factor from three boundary conditions: two conditions at the edge postulate continuity of elevation and current flux normal to the edge and one condition at the coast postulates vanishment of the normal component of current flux. This mathematical problem can be interpreted physically as such that amplitudes of stationary waves in the shelf and those of reflected waves in the open sea may be determined as the ratios to the incident waves.

When the waves in the open sea are of Kelvin wave type, there is only one elementary solution possible in the open sea representing waves propagating parallel to the coast. However, three boundary conditions must be satisfied as in case of the plane waves. These boundary conditions specify the value of k which becomes equal to $\sigma (gh)^{-\frac{1}{2}}$ if the open sea is bounded by a straight coast only (Lamb, 1945).

For simplicity a continental shelf with a uniform depth h_I is considered. The solution of equation (17) for the elevation over the shelf is expressed by:

$$\zeta_I = A e^{i m_I x} + B e^{-i m_I x} \tag{28}$$

and for the elevation in the open sea:

$$\zeta_{II} = E e^{-m_{II} x} \tag{29}$$

where m_I and m_{II} are defined by:

$$m_I^2 = (\sigma^2 - f^2)(g h_I)^{-1} - k^2 \tag{30a}$$

$$m_{II}^2 = -(\sigma^2 - f^2)(g h_{II})^{-1} + k^2 \tag{30b}$$

The boundary condition at the coast (x = 0) that $u_I = 0$ yields the relation

$$B/A = \mu = (i\sigma m_I + fk)(i\sigma m_I - fk)^{-1} \tag{31}$$

The boundary conditions that $u_I = u_{II}$ and $\zeta_I = \zeta_{II}$ at $x = l$ yield the equation for k:

$$h_I(fk + i\sigma m_I \mu_l) = h_{II}(fk - \sigma m_{II}) \tag{32}$$

in which $\mu_l = (e^{i m_I l} - \mu e^{-i m_I l})/(e^{i m_I l} + \mu e^{-i m_I l})$ (33)

If the depths of the shelf and the open sea are nearly equal, the presence of the shelf does not noticeabley affect the mode of waves in the open sea. Therefore, it is considered that the value of k satisfying (32) is nearly equal to $\sigma (g h_{II})^{-\frac{1}{2}} (= k_0)$ which satisfies this equation in case of no shelf. In fact, if the solution of k in (32) is expressed by

$$k = k_0 + \varkappa \tag{34}$$

it is easily seen that $\varkappa$ becomes a small quantity compared to k_0, by substituting (34) into (31) and (32). (Because μ^{-1} diminishes to zero if $i m_I$ is taken positive, the equation (32) becomes almost the same as in case of no shelf.)

More interesting is the case, in which the depth of the shelf is much less than that of the open sea, corresponding to an actual situation. This condition leads to the value of μ nearly equal to unity, if σ is not nearly equal to f. Further, if the width of the shelf is not so large, the effect of the shelf on the waves in the open sea is considered not to be conspicuous. Therefore, substitution of (34) in equation (32) yields, after a little manipulation:

$$\chi = \pi f(\sigma m_0 \tan m_0 l - k_0 f)\,[\sigma^2 - f^2(1-\pi)]^{-1} \quad (35)$$

where $$\pi = h_I / h_{II}\,;\quad m_0^2 = (\sigma^2 - f^2)(g h_I)^{-1} - \sigma^2 (g h_{II})^{-1} \quad (36a,\ b)$$

If we take a following numerical example:

$h_I = 100\,(m)$, $h_{II} = 2{,}000\,(m)$, $\sigma = 1.43\times 10^{-4}\,(sec^{-1})$
$f = \frac{1}{2}\sigma$, $l = 200\,(km)$

equations (34) to (36) yield

$k_0 = 10^{-6}\,(cm^{-1})$, $m_0 = 3.3\times 10^{-6}\,(cm^{-1})$, $\chi = 7\times 10^{-8}\,(cm^{-1})$

Therefore, the condition that χ is much less than k_0 is satisfied.

In the two-layered ocean, mathematical problems to be solved can be physically interpreted almost similarly to those in the ocean with a uniform density. If $k_m > k$ in the open sea, or the waves both of barotropic and baroclinic modes are of plane wave type, four elementary solutions representing incident and reflected waves of both modes are possible. The equation (14) yields also four elementary solutions over the shelf. Therefore, there are eight unknown coefficients to be determined from boundary conditions at the coast and at the edge of the open sea. The conditions at the coast are two in number, postulating vanishment of current flux in each layer normal to the coast. The conditions at the edge are four in number, postulating continuity of elevation of the upper surface and of flux normal to the edge in each layer. Thus, six unknown coefficients including amplitudes of reflected surface and internal waves can be determined from these boundary conditions as ratios to amplitudes of incident surface and internal waves: two supposedly known coefficients. Rattray's theory (1959) is a special case in this type of wave system in which the waves propagate normally to the coast. He assumed further that there is no incident internal waves. Therefore, in principle the six unknown coefficients can be expressed by the ratios to the amplitude of the incident surface tides. He assumed further that the width of the shelf is much less than the wave length of surface tides. Then the elementary solutions representing incident and reflected waves of the barotropic modes become constant in a range of x small compared to this wave length. He expressed two unknown coefficients representing amplitudes of internal seiches in the shelf and of reflected internal waves in the open sea in terms of the surface tides consisting of incident and reflected waves.

In the open sea, if $k > k_n$ or the waves of both modes are of Kelvin wave type, numbers of possible elementary solutions are two. In the shelf there are four elementary solutions as in case of $k < k_m$. The numbers of boundary conditions are six as before. These six boundary conditions yield six linear homogeneous equations on the six unknown constants. Therefore, in order to obtain non-trivial solutions, the determinant consisting of coefficients to the unknown constants in the six equations should vanish. This condition leads to the equation for k similar to equation (32) which is the result for a uniform ocean.

In the open sea if $k_n > k > k_m$, the waves of barotropic or baroclinic mode are respectively of Kelvin wave or plane wave type. If there are incident internal waves in the open sea, the numbers of elementary solutions are three. Since the numbers of elementary solutions on the shelf and of the boundary conditions are the same as in other cases of the two-layered ocean, six unknown coefficients can be determined as the ratios to one known coefficient, say, the amplitude of the incident internal waves. This model corresponds to a case when the internal waves oncoming from an infinite distance are reflected by the shelf. However, more practical is a problem in which the surface Kelvin waves in the open sea generate the internal waves outgoing from the shelf area. In this problem there are no incident internal waves in the open sea. Therefore, six boundary conditions yield six linear equations about six unknown coefficients as in case that $k > k_n$. The characteristic equation for k is obtained in order to avoid trivial solutions. The details of this case is discussed in the next chapter, since this seems to happen most commonly in an actual ocean.

5. Waves on a shelf with a uniform depth.

Physical features of the wave system corresponding to a case when $k_m < k < k_n$ in the open sea are clearly explained with a model of the simplest geometrical configuration; a shelf and an open sea of uniform depths and a constant depth of the upper layer. It is assumed that the internal waves are generated by the presence of a coast, eliminating incident internal waves in the open sea. The elementary solutions of barotropic and baroclinic modes in the open sea are given by

$$\zeta'_{s\,II} = E e^{-m_{II} x} \tag{37}$$

$$\zeta''_{i\,II} = F e^{-i n_{II} x} \tag{38}$$

and those in the shelf are:

$$\zeta'_{s\,I} = A e^{i m_I x} + B e^{-i m_I x} \tag{39}$$

$$\zeta''_{i\,I} = C e^{i n_I x} + D e^{-i n_I x} \tag{40}$$

where suffices I and II refer to the shelf and the open sea, respectively. Since ζ_i' is smaller than ζ_s' by one to two orders of magnitude, the elementary solutions of barotropic mode are represented in terms of ζ_i'' instead of ζ_i'. In equations (37) to (40), m_{II} and n_{II} must be real and positive, while m_I and n_I may be either real or imaginary.

Boundary conditions at the coast (x = 0) are

$$u_I' = 0 \quad ; \quad u_{II}'' = 0 \qquad \text{(41a) (41b)}$$

Since $u' = u_s' + u_i'$ and $u'' = u_s'' + u_i''$, boundary conditions (41a) and (41b) with (22) and (23) yield

$$u_{iI}' = u_{sI}' = u_{iI}'' = u_{sI}'' = 0 \quad (at\ x=0) \qquad (42)$$

Substituting (39) and (40) into (7) and (9) with conditions (42) yields:

$$B/A = \mu = (i\sigma m_I + fk)/(i\sigma m_I - kf) \qquad (43)$$

$$D/C = \nu = (i\sigma n_I + fk)/(i\sigma n_I - kf) \qquad (44)$$

where the ratios B/A and D/C are designated by μ and ν, respectively.

Continuity of elevation of the free surface at the edge of the shelf (x = ℓ) yields (45) with an approximation $\zeta' \approx \zeta_s'$. When equation (19) is used, continuity of elevation of the interface at the edge yields

$$\left(Ee^{-m_{II}\ell} = Q = A(e^{im_I\ell} + \mu e^{-im_I\ell}) \qquad (45) \right)$$

$$\begin{aligned} r_I Q + C\,(e^{in_I\ell} + \nu e^{-in_I\ell}) \\ = r_{II} Q + F e^{-in_{II}\ell} \end{aligned} \qquad (46)$$

where

$$r_I = h_I''/h_I \quad ; \quad r_{II} = h_{II}''/h_{II} \qquad (47)$$

Continuity at x = ℓ of current fluxes normal to the edge of the shelf yields respectively in the upper and lower layers:

$$u_{sI}' + u_{iI}' = u_{sII}' + u_{iII}' \qquad (48)$$

$$h_I''(u_{sI}'' + u_{iI}'') = h_{II}''(u_{sII}'' + u_{iII}'') \qquad (49)$$

Substituting relations (22) and (23) into (49) and eliminating u_i' by use of (48), we have:

$$h_I u'_{sI} = h_{II} u'_{sII} \tag{50}$$

instead of (49). The boundary conditions (50) and (45) yield the same equation for k as equation (32) by eliminating A, B, and E of ζ'_{sI} and ζ'_{sII} . This relation (32) yields the proper value of k as in case of a homogeneous ocean. For this value of k , the amplitudes of internal waves C and F can be determined in terms of E or Q from the boundary conditions (46) and (48).

Equation (48) leads to

$$\begin{aligned} &n_I (k\nu_1 + i\sigma n_I \nu_2) C - n_{II}(fk - i\sigma n_{II}) F e^{-in_I l} \\ &= \sigma Q \delta^{-1} (m_{II} - m_I \tan m_I l) \end{aligned} \tag{51}$$

where

$$\nu_1 = e^{in_I l} + \nu e^{-in_I l} \tag{52}$$

$$\nu_2 = e^{in_I l} - \nu e^{-in_I l} \tag{53}$$

Eliminating F from (46) and (51) we have:

$$\begin{aligned} &[n_{II}\nu_1(kf - i\sigma n_{II}) - n_I(fk\nu_1 + i\sigma n_I \nu_2)] C \\ &= [(n_{II} - n_I) n_{II}(fk - i\sigma n_{II}) - \sigma\delta^{-1}(m_{II} - m_I \tan m_I l)] Q \end{aligned} \tag{54}$$

The assumptions which were made in deriving equation (35) lead to the following relations among constant s included in equation (54).

$$k \approx \sigma (g h_{II})^{-\frac{1}{2}} (= k_0) \tag{55}$$

$$fk \ll \sigma n_I \; ; \; fk \ll \sigma n_{II} \tag{56a,b}$$

$$n_I \approx \left[\frac{h_I(\sigma^2 - f^2)}{g' h' h_I''}\right]^{\frac{1}{2}} (= k_1) ; n_{II} = \left[\frac{h_{II}(\sigma^2 - f^2)}{g' h' h_{II}''}\right]^{\frac{1}{2}} (= k_2) \tag{57a.b}$$

$$m_I^2 \approx \frac{\sigma^2 - f^2}{g h_I} - \frac{\sigma^2}{g h_{II}} (= m_1^2) ; m_{II}^2 \approx \frac{f^2}{g h_{II}} (= m_2^2) \tag{58a,b}$$

With these relations (55) to (58) equation (54) can be written as:

$$2C\left[\cos k_1 l + i\,(R_I/R_{II})^{\frac{1}{2}} \sin k_1 l\right] = Q\,(R_{II} - R_I - i\psi) \qquad (59)$$

where

$$\psi = (\delta R_{II} k_2)^{-1}(m_2 - m_1 \tan m_1 l) \qquad (60)$$

These approximation formulas yield the amplitude of reflected internal waves as:

$$F e^{-ik_2 l}\left[\cos k_1 l + i\,(R_I/R_{II})^{\frac{1}{2}} \sin k_1 l\right] = -i\left[(R_{II} - R_I)(R_I/R_{II})^{\frac{1}{2}} \sin k_1 l + \psi \cos k_1 l\right] Q \qquad (61)$$

The relation (56a) leads to an approximation $\nu \approx 1$.
The amplitude of the internal seiches in the shelf is given by 2C. In order to compare with the result of Rattray (1959), the amplitude of surface Kelvin waves Q is assumed to be equal to that of the surface tides of his theory. If <u>Ar</u> is the amplitude of internal seiches of his theory, the ratio 2C/Ar is given by:

$$2C/A_r = \phi = \frac{R_{II} - R_I - i\psi}{(R_{II} - R_I)(1 + i k_2 l)} \qquad (62)$$

In Figure 2 the curve of $|\phi|$ against the ratio σ/f is plotted. The same values of depths of both layers, δ and f as listed in Table 1 are used. The width of the shelf l is equal to 100 km. The curve indicates that internal seiches due to the surface Kelvin waves become much larger than those due to the plane waves propagating normal to the coast as the period approaches to one pendulum day. This situation is more clearly illustrated in Figure 3, which shows the amplitudes (ratios to the surface tides) of the internal seiches and outgoing internal waves in the open sea as functions of $l k_1 R_I^{\frac{1}{2}} = l k_2 R_{II}^{\frac{1}{2}}$ for frequency $\sigma = 1.1 f$ and $\sigma = 2f$, respectively. For the surface Kelvin waves, 2C/Q and F/Q are determined respectively from (59) and (60), and for the plane waves, equations (34) and (35) in Rattray's (1959) paper are used, The same physical and geometrical constants as Figure 2 are used. It is seen that both internal seiches and internal reflected waves generated by the surface Kelvin waves become conspicuous for the frequency $\sigma = 1.1f$.

6. Internal waves with a period of a pendulum day.

In case that $\sigma = f$, equations (7) to (10) are reduced to two equations:

$$\left(\frac{d}{dx} + k\right)\zeta' = 0 \; ; \quad \left(\frac{d}{dx} + k\right)\zeta'' = 0 \qquad \text{(63a,b)}$$

The solutions of these equations are:

$$\zeta' = M e^{-kx} \; ; \quad \zeta'' = N e^{-kx} \qquad \text{(64a,b)}$$

If the sea is assumed to extend from $x = 0$ to positive $\underline{x}$, the waves which are finite at an infinite distance must propagate to the direction of negative $\underline{y}$, (Kelvin waves) because $\underline{k}$ should be positive in (64a)and(b). Substituting (64a)and(b) into (1), (3), (5) and (6) and eliminating $\underline{v'}$ and $\underline{v''}$, we have:

$$\frac{d}{dx}(h'u') - kh'u' = i[gk^2h'\sigma^{-1}M + \sigma(N-M)]e^{-kx} \qquad (65)$$

$$\frac{d}{dx}(h''u'') - kh''u'' = i[k^2h''\sigma^{-1}(g''M + g'N) - \sigma N]e^{-kx} \qquad (66)$$

Boundary conditions at $x = 0$ are vanishment of $\underline{h'u'}$ and $\underline{h''u''}$. Integrating (66) and (67) with these conditions, we have:

$$h'u' = ie^{kx}\left[\sigma(N-M)(1-e^{-2kx})/(2k) + gk^2\sigma^{-1}M\int_0^x h'e^{-2kx}dx\right] \qquad (67)$$

$$h''u'' = ie^{kx}\left[\sigma N(e^{-2kx}-1)/(2k) + k^2\sigma^{-1}(g''M + g'N)\int_0^x h''e^{-2kx}dx\right] \qquad (68)$$

The boundary conditions that $h'u'$ and $H''u''$ are finite at $x =$ postulate the relations:

$$\sigma^2(N-M) + 2gk^3M\int_0^\infty h'e^{-2kx}dx = 0 \qquad (69)$$

$$-\sigma^2N + k^3(g''M + g'N)\int_0^\infty h''e^{-2kx}dx = 0 \qquad (70)$$

Equations (69) and (70) yield the characteristic equation of k for non-trivial constant M and N.

If the shelf and the open sea are assumed to have uniform depths as in the previous chapter, equations (70) and (71) become:

$$(\sigma^2 - g k^2 h')M - \sigma^2 N = 0 \tag{71}$$

$$k^2 (g'' M + g' N)(h_I'' - h_I'' e^{-2kl} + h_{II}'' e^{-2kl}) - \sigma^2 N = 0 \tag{72}$$

Elimination of M and N from (72) and (73) yields:

$$g k^2 [h_I + (h_{II}'' - h_I'') e^{-2kl}] - g g' h' \sigma^{-2} k^4 [h_I'' + (h_{II}'' - h_I'') e^{-2kl}] - \sigma^2 = 0 \tag{73}$$

This transcendent equation for k has roots of barotropic and baroclinic modes. For the barotropic mode, k satisfying (73) is nearly equal to $\sigma (g h_{II})^{-\frac{1}{2}} (=k_0)$ and yields $kl \ll 1$. The second approximation of k is obtained from (73) and N/M is given by (71).

$$k \approx \sigma [g \{h_I + (h_{II}'' - h_I'') e^{-2k_0 l}\}]^{-\frac{1}{2}} \tag{74}$$

$$N/M \approx (1 - h'/h_{II}) \tag{75}$$

Equation (75) is the same as equation (19) which was derived with the assumption that $f \neq \sigma$. For the baroclinic mode, the value of k satisfying (73) leads to $kl \gg 1$. Equations (74) and (72) yield respectively k and N/M:

$$k \approx \sigma [h_I / g' h' h_I'']^{\frac{1}{2}} \tag{76}$$

$$N/M \approx (1 - \delta^{-1} h_I / h_I'') \tag{77}$$

Equation (77) is essentially the same as (20). Equation (76) indicates that the internal Kelvin waves in this system must have the wave velocity of internal waves on the shelf with no rotation.

Fig. 2 Amplitude ratio of Ichiye's to Rattray's internal waves (outgoing) against σ/f.

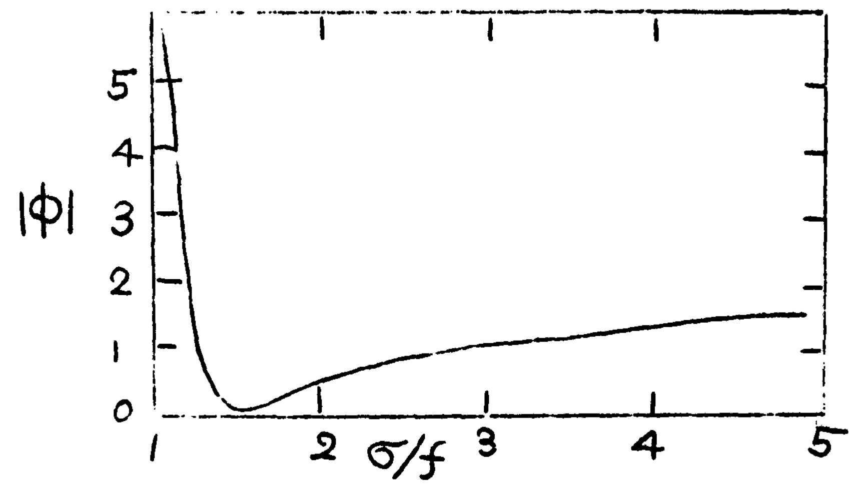
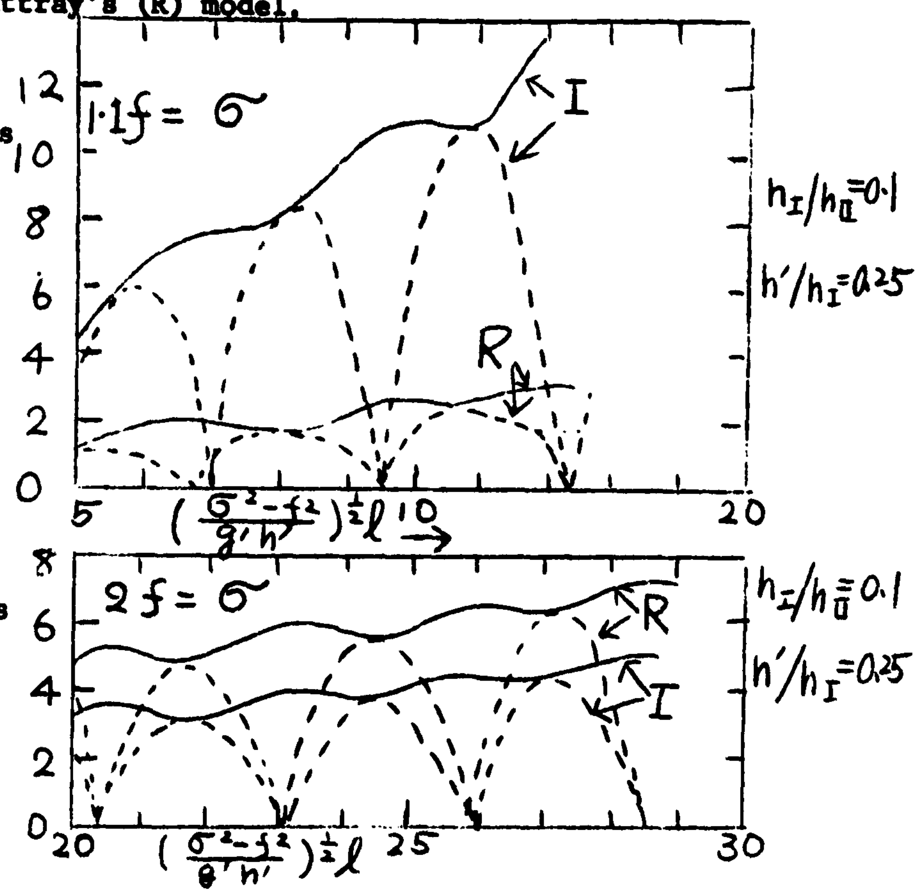


Fig. 3 Amplitudes of internal seiches (full lines) and outgoing internal waves (broken line) for Ichiye's (indicated by I) and Rattray's (R) model.

Amplitude expressed as ratio to that of incident surface waves.



Amplitude expressed as ratio to that of incident surface waves.

7. Summary and Application.

The wave system in a two-layered rotating ocean bounded with a shelf having a straight coastline and a constant width can be summarized according to relative magnitude of its frequency σ and the Coriolis' coefficient $\underline{f}$.

(1) $\sigma > f$. The wave numbers $\underline{k}$, $\underline{km}$, and $\underline{kn}$ indicate those in the direction of the coast, barotropic and baroclinic mode of the open sea, respectively. (a) $k < km$. Both modes are plane waves propagating in different directions (the barotropic waves with a larger angle to the coast). (b) $kn > k > km$. Barotropic and baroclinic modes are respectively Kelvin waves and plane waves, $\underline{k}$ corresponds to barotropic Kelvin waves. (c) $k > kn$. Both modes are Kelvin waves, $\underline{k}$ corresponds to internal Kelvin waves.

(2) $\sigma \ll f$. Both barotropic and baroclinic modes are Kelvin waves. The characteristic equation yields the values of $\underline{k}$ almost separately for barotropic and baroclinic modes.

Energy relationships between incident and reflected waves of both modes may be discussed in a similar manner to Snodgrass and others (1962). However, the primary purpose of this work is to suggest a qualitative explanation on the observed temperature fluctuation of tidal periods at an offshore platform in the Gulf of Mexico near Panama City, Florida (Salsman, 1962). The BT records indicate that the oscillations of the thermocline have amplitudes reaching as great as 30 feet with a period of diurnal tides during tropic tides, but that these oscillations are much smaller in amplitudes and irregular in periods during equatorial tides. A pendulum day on the northern Gulf Coast is nearly equal to one day. The surface tides of the area considered are diurnal during tropic tides, but of mixed type during equatorial tides. However, tidal ranges during the latter almost reach half of those during the former. Therefore, if the amplitudes of internal tides are simply proportional to those of the surface tides, the oscillations of the thermocline during equatorial tides shou reach half of those during tropical tides. Also Rattray's (1959) theory indicates that the amplitudes of the internal waves decrease as the period approaches to a pendulum day. His theory is based on an assumption that the surface tides propagate perpendicularly to the coast. However, a cotidal chart of the Gulf of Mexico suggests that the tidal waves move parallel to the northern coast. Therefore, Rattray's theory is not applicable to the area considered. The result obtained in Chapter 5 suggests that if the surface tides propagate parallel to the coast as Kelvin waves, diurnal tides may cause much larger internal tides than semi-diurnal ones owing to the condition that a pendulum day there is close to one day. The result of Chapter 6 postulates that if the period is exactly equal to a pendulum day, large internal tides may be generated only when the surface tides have a wave length nearly equal to that of the internal Kelvin wave. The surface tides may have components of small magnitudes with such wave length because amplitudes of 1 cm can generate internal waves of about 10 m according to equation (77). Also, see Summers and Emery (1963) for observations on waves normal to the coast.

Acknowledgements:

This work is an outcome of frequent discussions with George B. Dowling and his group of U. S. Navy Mine Defense Laboratory and Roy D. Gaul and his group at the A and M College of Texas. The author is indebted to those who were tempted by him to test (or detest) his theory by wetting their feet. Their tedious field works are still going on and in due time their results will support (or smash) some of the arguments presented here. The author also wishes to thank Mrs. Suzy Bowman for typing the manuscript.

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For more complete bibliography on internal waves, see:

"Annoted Bibliography on Internal Waves" by T. Ichiye. Technical Paper No. 1, July 1962.

Errata

Page 3 line 2. Read $-6(\sigma^2 - f^2)\zeta''$ for the r.h.s. of equation (12).
Substitute equation (13) for (12) and vice versa.